The E_8 Lattice, Sphere Packing And The Theory Of Everything !?!

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Overview of the presentation:

1. Define the sphere packing problem for $\mathbb{R}^d$.
2. Give a brief historical overview of how E_8 and other exceptional Lie Algebras were first encountered.
3. Interlude on lattices, theta functions other exceptional properties of E_8 and Λ_{24}.
4. Sphere packing in dimensions 8 and 24, universal optimality.

Sphere packing in dimensions $d \geqslant 1$:

Question : What is the densest packing of congruent balls in $\mathbb{R}^d$,, i.e. the packing which covers the largest proportion of a given euclidean space?

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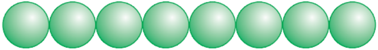
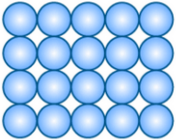
Answer : This is a question with astounding depth and in almost every dimension d we have no clue.

Sphere packing in dimensions $d \geqslant 1$:

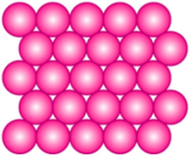
- In dimension 1 the problem is trivial. We have 100% density.



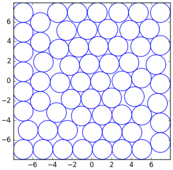
- In dimension 2, Thue [**1**] proved in 1890 that the densest packing had density $\frac{\pi}{\sqrt{12}}$ which is approx. 90.6%.



Square packing

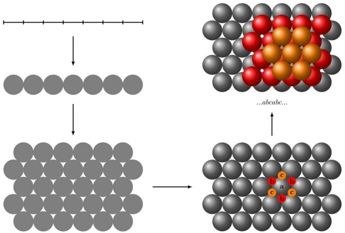


Hexagonal packing



Sphere packing in dimensions $d \geqslant 1$:

- In 1611, Johannes Kepler conjectured that the optimal packing came from face centered cubic lattice which covers a proportion of $\frac{\pi}{\sqrt{18}}$ of $\mathbb{R}^3$ (or just around 74%).
- Intuitively though..



Sphere packing in dimensions $d \geqslant 1$:

- Only in 2005 was the Kepler conjecture proven by Thomas Hales [**2**] using a computer assisted proof.
- Formally verified in 2017 by a team of 22 people.

Sphere packing in dimensions $d \geqslant 1$:

What about in dimension $d > 3$?

- Geometry becomes much less intuitive.
- Take the unit cube in $\mathbb{R}^{1000}$. It has volume 1, but what if we reduce the side length by 1 percent? Well,

$$0.99^{1000} = 0.0000431712474107$$

- What about in $\mathbb{R}^{10000}$?

$$0.99^{10000} = 2.24877484982 \cdot 10^{-44}$$

Sphere packing in dimensions $d \geqslant 1$:

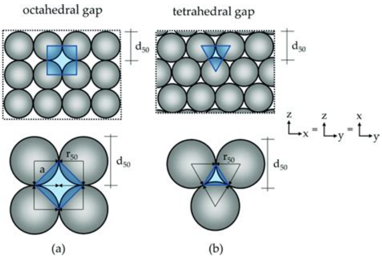
Question : Why should the densest packings come from highly symmetric objects such as lattices?

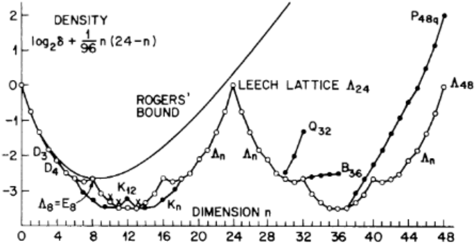
"In ten dimensions the best packing known is the Best packing, which is not a lattice packing. "

"By the way, not only are the best current lower bounds the same for lattices and non-lattice packings in high dimensions, but so are the upper bounds, so asymptotically we are totally unable to distinguish between these cases. Still, just about everyone believes lattices must be worse eventually. A lattice in $\mathbb{R}^n$ is determined by a quadratic number of parameters, but there are an exponential number of gaps to fill, so it's just not plausible that you can do well when n is huge. (However, there's little hope of making this sort of argument rigorous.) - Henri Cohn, MathSE"

Sphere packing in dimensions $d \geqslant 1$:

How do we fill these gaps?





Sphere packing in dimensions $d \geqslant 1$:

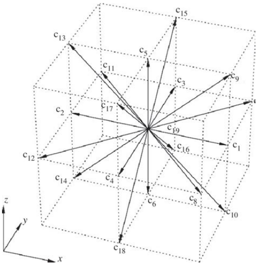
Upshot : Sphere packing has been resolved in dimensions 8 and 24 in which the densest packings were come from objects relating to the E_8 lattice and Leech lattice Λ_{24} .

- Announced on pi day of 2016, a proof [**3**] from Prof. Maryna Viazovska in $d = 8$.
- Later proven in $d = 24$ along with H. Cohn, A. Kumar, S. D. Miller, D. Radchenko (stronger result).
- 2022 Fields medal winner.

Prelude : Lattices

For us, we say that an integral lattice L is a free abelian group of finite rank together with a symmetric bilinear form f.

$$L = \bigoplus_{i=1}^{n} \mathbb{Z} \quad f : L \times L \to \mathbb{Q} \quad f(x, y) = f(y, x) \quad \forall x, y$$



Examples: $\mathbb{Z}^n$, $\mathbb{Z}[\omega]$

Prelude : Lattices

- Positive definite: $(a, a) \geqslant 0$ for all $a \in L$.
- Even : (a, a) is an even number for all $a \in L$.
- Unimodular : Determinant of L is ± 1. Equivalently, L^* is integral, so we have $L = L^*$. Also called self-dual. This gives rise to nice propeties.

What came first: The Lie Algebras, Their Classification or the Lattices?

"For each ℓ there are four structures supplemented for ℓ=2,4,6,7,8 by exceptional simple groups. For these exceptional groups I have various results that are not in fully developed form; I hope later to be able to exhibit these groups in simple form and therefore am not communicating the representations for them that have been found so far." - Killing, 1887

What came first: The Lie Algebras, Their Classification or the Lattices?

A bit about the E_8 Lie group.

- Dimension 248, Rank 8.
- Order of the Weyl group is $4!6!8! = 696729600$. Many symmetries.
- Unique among simple compact lie groups. Its first non trivial representation has dimension 248 and is the adjoint representation.
- Trivial centre, compact, simply connected, all roots have equal length, trivial automorphism group.

What came first: The Lie Algebras, Their Classification or the Lattices?

A bit about the E_8 Lie group.

- Constructed from the automorphism group of the Lie algebra $\mathfrak{e}_8$.
- This contains $\mathfrak{so}(16)$ generated by 120 J_{ij}'s as a sub-algebra.
- Also contains **spin**(16) generated by 128 Q_a's called spinors.

$$[J_{ij}, J_{kl}] = \delta_{jk} J_{il} - \delta_{jl} J_{ik} - \delta_{ik} J_{jl} - \delta_{il} J_{jk}$$

$$[Q_a, Q_b] = \frac{1}{8} \sum_{i,j} |\gamma^i, \gamma^j|_{ab} J_{ij}$$

$$[J_{ij}, Q_a] = \frac{1}{4} \sum_{b} |\gamma_i, \gamma_j|_{ab} Q_b$$

What came first: The Lie Algebras, Their Classification or the Lattices?

Now, about the E_8 root system. As per humphreys, we have $I' = \mathbb{Z}^8 + (\mathbb{Z} + 1/2)^8$, I'' is the subset with elements of form

$$\sum_{i=1}^{8} c_i e_i + \frac{c}{2}(e_1 \cdots e_8)$$

such that $c + \sum c_i$ is even. Then $\Phi = \{a \in I'' \mid (a, a) = 2\}$. All of the root vectors are either of form $\pm(e_i \pm e_j)$ for $i \neq j$ ($2^2\binom{8}{2} = 112$ of these) or $(\pm\frac{1}{2}, \pm\frac{1}{2}, \cdot, \pm\frac{1}{2})$ where the total number of negative signs is an even integer (2^7=128 of these).

From this we can read off integrality and even-ness of the root lattice. Being unimodular can be checked readily.

What came first: The Lie Algebras, Their Classification or the Lattices?

Dimension	Odd lattices	Odd lattices no roots	Even lattices	Even lattices no roots
0	0	0	1	1
1	1	0		
2	1	0		
3	1	0		
4	1	0		
5	1	0		
6	1	0		
7	1	0		
8	1	0	1 (E_8 lattice)	0
9	2	0		
10	2	0		
11	2	0		
12	3	0		
13	3	0		
14	4	0		
15	5	0		
16	6	0	2 (E_8^2, D_{16}^+)	0
17	9	0		
18	13	0		
19	16	0		
20	28	0		
21	40	0		
22	68	0		
23	117	1 (shorter Leech lattice)		
24	273	1 (odd Leech lattice)	24 (Niemeier lattices)	1 (Leech lattice)
25	665	0		
26	≥ 2307	1		
27	≥ 14179	3		
28	≥ 327972	38		
29	≥ 37938009	≥ 8900		
30	≥ 20169641025	≥ 82000000		
31	≥ 5000000000000	≥ 800000000000		
32	≥ 80000000000000000	≥ 10000000000000000	≥ 1160000000	≥ 10900000

Takeaway: Even, unimodular lattices are rare for small dimensions.

What came first: The Lie Algebras, Their Classification or the Lattices?

To answer the question from before...

- The existence of an even unimodular lattice in $\mathbb{R}^8$ was first proven by H.J.S. Smith in 1867 and followed from his newly discovered mass formula for lattices. When d is divisible by 8 :

$$\sum_{\Lambda} \frac{1}{|Aut(\Lambda)|} = \frac{|B_{d/2}|}{d} \prod_{1 \leqslant j < d/2} \frac{|B_2 j|}{4j}$$

- The mass in dimension 8 is

$$\frac{|-1/30|}{8} \frac{|1/6|}{4} \frac{|-1/30|}{8} \frac{|1/42|}{12} = \frac{1}{696729600}$$

- Such a lattice exists!
- Only constructed in 1873 by Korkin and Zolotarev.

What came first: The Lie Algebras, Their Classification or the Lattices?

Another reason why even unimodular lattices are nice.

- To any lattice Λ you can associate a "Theta function" in the following way:

$$\Theta_\Lambda(z) = \sum_{x\in\Lambda} e^{2i\pi z|x|^2} \quad Imz > 0$$

- Holomorphic functions on $\mathbb{H}$.
- If Λ is even and unimodular and of rank n, we get a modular form of weight $n/2$.

$$f(\gamma z) = (cz+d)^k f(z) \quad \forall\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

What came first: The Lie Algebras, Their Classification or the Lattices?

Another reason why even unimodular lattices are nice.

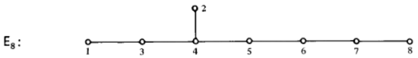
- For $d = 8$, we would recover a modular form of weight 4 (level 1) and we know that the space of modular forms $M_4(\Gamma(1))$ is one dimensional and spanned by the eisenstein series $G_4(z)$. In general for $k \geqslant 4$ we have

$$E_k(z) = 1 + \frac{2}{\zeta(1-k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^z$$

- Thus, we recover $G_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^z$.
- Looking at the Fourier expansion $G_4(z) = 1+240q^2+2160q^4+6720q^6+17520q^8+30240q^{10}+60480q^{12}+\cdot$ we see that this counts exactly how many lattice points have norm n.

About the geometry of E_8

- "The E8 lattice points are the vertices of the 5_{21} honeycomb, which is composed of regular 8-simplex and 8-orthoplex facets. With the following Coxeter-Dynkin diagram:"



- Highly regular, W acts transitively on the k-faces for $k \leqslant 6$. All of the k-faces are simplices.
- A hole in a lattice is a point in the ambient Euclidean space whose distance to the nearest lattice point is a local maximum.

About the geometry of E_8

- Each point of the E_8 lattice is surrounded by 2160 8-orthoplexes and 17280 8-simplices. The 2160 deep holes near the origin are exactly the halves of the norm 4 lattice points. The 17520 norm 8 lattice points fall into two classes (two orbits under the action of the E_8 automorphism group): 240 are twice the norm 2 lattice points while 17280 are 3 times the shallow holes surrounding the origin.

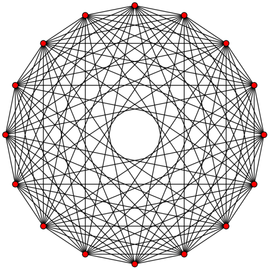


Figure: 8-orthoplex projection

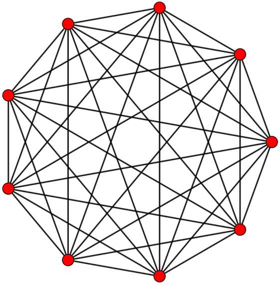


Figure: 8-simplex projection

About the geometry of E_8

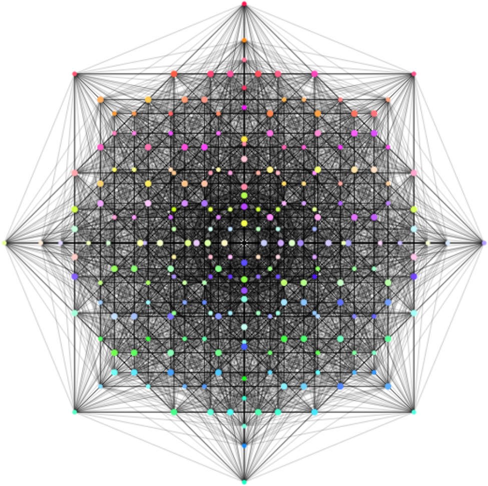


Figure: Projection of E_8

About the geometry of E_8

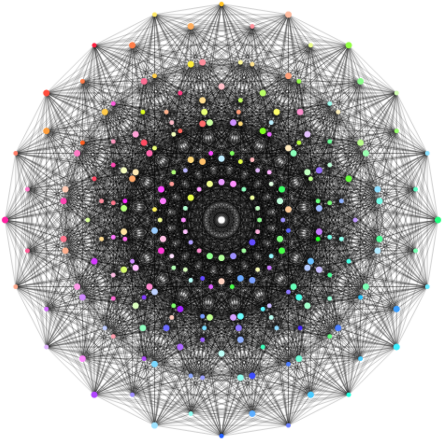


Figure: Projection of E_8 with 10 24-gons

About the geometry of E_8

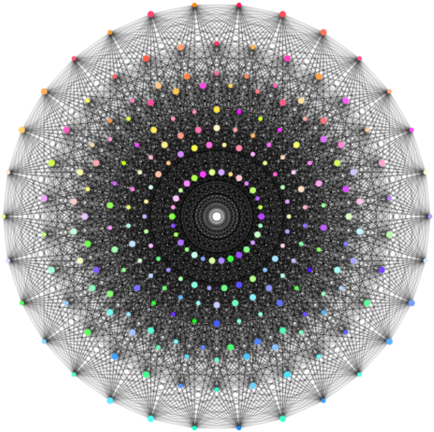


Figure: Projection of E_8 with 12 20-gons

Hamming Codes - Math in the real world??

In Computer Science and Telecommunications Hamming codes are a type of error correcting code.

- $H(n, k)$ is a binary code of length n and rank k. Lives in some k dimensional subspace of $\mathbb{F}_2^n$.
- Possible to construct a lattice from a binary code C of length n by taking the set of all $x \in \mathbb{Z}^n$ such that $x \equiv C \bmod 2$ for some codeword C.
- $H(8, 4) =$ $\{00, 0F, 33, 3C, 55, 5A, 66, 69, 96, 99, A5, AA, C3, CC, F0, FF\}$.

Hamming Codes - Math in the real world??

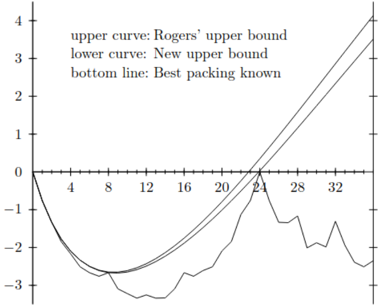
In Computer Science and Telecommunications Hamming codes are a type of error correcting code.

- Type 2 self-dual.
- Applying the previous construction to type 2 self dual $\implies$ even, unimodular lattice.
- $H(8,4)$ is isomorphic to E_8 (non-trivial).

Back to Sphere Packing

Given a lattice Λ, if the minimal distance between any two points is say ℓ, then we can optain a packing by placing unit balls with centers at $\frac{1}{\ell}\Lambda$. In the case of E_8 the minimal distance is $\sqrt{2}$ and so our lattice packing is $\frac{1}{\sqrt{2}}E_8$.

- Density of $\frac{\pi^4}{2^4 4!}$ which is approx. 0.253.
- Cohn and Elkies [**6**] introduce the idea of Linear programming bounds



Back to Sphere Packing

- Inspired from the proof of optimality of $H(8,4)$.
- Reduce geometric optimization problem on a space to minimizing a linear functional on a certain some cone of functions on of that space.
- Theorem [Cohn, Elkies] : suppose that $f : \mathbb{R}^d \to \mathbb{R}$ is integrable, non-zero and satisfies
  1. $f(x) \leqslant 0$ fo all $|x| \geqslant 1$
  2. $\hat{f}(t) \geqslant 0$ for all t.

  then the center density of an n-dimensional sphere packing is bounded above by
  $$\frac{f(0)}{2^n \hat{f}(0)}$$
- f can be assumed Shwarz, radial, radius can be altered.
- They wrote code which optimized over (evidently) finite spaces.

Back to Sphere Packing

By making certain assumptions about the problem, one could arrive at certain extra conditions which such a "magic function" would satisfy. Maryna carefully crafted this magic function which is

$$g(x) = \frac{\pi i}{2160} a(x) + \frac{i}{240} b(x)$$

where

$$a(x) := \int_{-1}^{i} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\, e^{\pi i \|x\|^2 z}\, dz + \int_{1}^{i} \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\, e^{\pi i \|x\|^2 z}\, dz \tag{35}$$

$$-2\int_{0}^{i} \phi_0\Big(\frac{-1}{z}\Big)\, z^2\, e^{\pi i \|x\|^2 z}\, dz + 2\int_{i}^{i\infty} \phi_0(z)\, e^{\pi i \|x\|^2 z}\, dz.$$

For $x \in \mathbb{R}^8$ define

$$b(x) := \int_{-1}^{i} \psi_T(z)\, e^{\pi i \|x\|^2 z}\, dz + \int_{1}^{i} \psi_T(z)\, e^{\pi i \|x\|^2 z}\, dz \tag{52}$$

$$-2\int_{0}^{i} \psi_I(z)\, e^{\pi i \|x\|^2 z}\, dz - 2\int_{i}^{i\infty} \psi_S(z)\, e^{\pi i \|x\|^2 z}\, dz.$$

Universal optimality of E_8 and the Leech lattices

Shortly thereafter, Maryna, Cohn, Kumar, Miller and Radchenko proved a much stronger result. The showed that these lattices are what is called *universally optimal*.

Def: Let C be a discrete subset of $\mathbb{R}^d$ with density $\rho > 0$. We say that C is universally optimal if it minimizes ρ-energy whenever $\rho : \mathbb{R}^+ \to \mathbb{R}$is a completely monotonic function of squared distance.

From this we immediately get sphere packing, the kissing number problem and other optimization problems in dimensions 8 and 24.

Thanks! :)

[1] A. Thue,
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[2] T. C. Hales,
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[3] M. S. Viazovska,
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[4] H. Cohn, A. Kumar, S. D. Miller, D. Radchenko, and M. S. Viazovska,
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[5] http://www.madore.org/ david/math/e8w.html

[6] H. Cohn, N. Elkies,

New upper bounds on sphere packings I, Ann. of Math. (2) 157 no. 2 (2013).