GENERAL STRUCTURE OF MELLIN TRANSFORMS

ANDREAS HATZIILIOU AND ARNAB BOSE

Abstract. In this article, we give a brief introduction to the general structure of Mellin transforms and their inverses. We discuss the context in which these objects came about as well as the significant role they play in number theory and other areas of mathematics.

1. History of Mellin Transforms

The Mellin transform is a mathematical operation that generalizes the Fourier transform to functions of a more general form. It was introduced by the Finnish mathematician R. J. Mellin in 1897 in his paper "Zur Theorie zweier allgemeinen Klassen bestimmter Integrale" [R.J97]. While other integral transforms was typically introduced to tackle physical problems, Mellin transforms arose from a more mathematical context. $\int_0^\infty$

Definition. *The Mellin transform $\{\mathcal{M}f\}(s)$ of a function $f(x)$ is defined as an integral transform of the form:*

$$\{\mathcal{M}f\}(s) = \varphi(s) = \int_0^\infty x^{s-1} f(x)dx, \tag{1.1}$$

whose boundaries are determined by the analytic structure of $f(x)$ as x goes to 0 and infinity. If we suppose that the function $f(x)$ is $O(x^{-a-\varepsilon})$ as $x \to 0$ and $O(x^{-b+\varepsilon})$ as $x \to \infty$ for $\varepsilon > 0$ and $a < b$, then the integral above converges absolutely and defines an analytic function on $a < \Re(s) < b$, called the "strip of analycity" of $\{\mathcal{M}f\}(s)$.

Example. *Let $f(t) = H(t - t_0)t^z$, where H is the Heaviside step function, t is a positive number and z is a complex number. The Mellin transform of f is*

$$\{\mathcal{M}f\}(s) = \varphi(s) = \int_{t_0}^\infty t^{z+s-1} dt = -\frac{t_0^{z+s}}{z+s},$$

where s is such that $\Re(s) < -\Re(z)$. Here, $f(s)$ is holomorphic in a half-plane.

Similar to the Fourier transform, the Mellin transform provides a way to represent a function in a different domain, which can be useful in various areas of mathematics and physics. In fact, both Fourier and Laplace transforms can be derived from (1.1) by a change of variables. Setting $x = e^{-t}$ and hence $dx = -e^{-t}dt$, (1.1) becomes

$$\{\mathcal{M}f\}(s) = \int_{-\infty}^\infty f(e^{-t})e^{-st} dt = \{\mathcal{L}f(e^{-t})\}(s), \tag{1.2}$$

where $\mathcal{L}$ denotes the two-sided Laplace transform. If we let $a, b \in \mathbf{R}$ and setting $s = a + 2\pi ib$ in (1.2), we have

$$\{\mathcal{M}f\}(s) = \int_{-\infty}^{\infty} f(e^{-t})e^{-at}e^{-2\pi ibt}dt = \{\mathcal{F}f(e^{-t})e^{-at}\}(b),$$

where $\mathcal{F}$ denotes the Fourier transform.
As we will see later, the Mellin transform inherits many similar properties to the Laplace transform. But first, we note that its inversion formula follows similarly as in the case of the Laplace Transform:

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\varphi(s)\, ds,$$

where $c \in \mathbb{C}$ lies to the right of all singularities of $\{\mathcal{M}f\}(s)$. As also mentioned in [PK01], this type of integral inversion formula was known to Riemann, who famously used these ideas in [Rie53] which led to the proofs of the prime number theorem - as well as earlier to de la Vallée Poussin, but was only given rigourous justification by Mellin almost a century later.

2. Properties and cutoff functions

We begin this section by noting a few translation properties that the Mellin transform possess. For any proofs, we refer the reader to [Sne72], although these are all quite straightforward.

Proposition 2.1. *Let $a, b > 0$ and $n \in \mathbb{N}$, let $\varphi(s) = \int_0^\infty x^{s-1}f(x)\, dx$. Then, the following hold.*

$$\begin{aligned}
&\{\mathcal{M}f(ax)\}(s) = a^{-s}\varphi(s)\\
&\{\mathcal{M}x^a f(x)\}(s) = \varphi(s+a)\\
&\{\mathcal{M}f(x^a)\}(s) = a^{-1}\varphi(s/a)\\
&\left\{\mathcal{M}f(x^{-a})\right\}(s) = a^{-s}\varphi(-s/a)\\
&\left\{\mathcal{M}x^a f(x^b)\right\}(s) = b^{-1}\varphi(-(s+a)/b)\\
&\{\mathcal{M}(\ln x)^n f(x)\}(s) = \varphi^{(n)}(s)
\end{aligned}$$

Theorem. *Suppose the complex valued function f is of the form $f = \sum_{k=0}^{\infty}\frac{\phi(x)}{k!}(-x)^k$ for some integrable function ϕ. Then the Mellin transform of f is given by*

$$\{\mathcal{M}f(x)\}(s) = \Gamma(s)\phi(-s)$$

This powerful tool which was known to Ramanujan, who wrote it down in one of his notebooks, is the key in calculating various integral and series. It leads to many important integral representations such as those of Bessel functions as well as identities involving the Gamma function. G. H. Hardy even stated in [Har02] that he "was particularly fond of them" and used them as one of his commonest tools."

Example. *The Bessel Function $J_\nu(x)$ is given by*

$$J_\nu(x) = \frac{1}{\Gamma(\nu+1)} \frac{x^\nu}{2^\nu} {}_0F_1(-;\nu+1;\frac{-x^2}{4})$$

where ${}_pF_q(c;d;-x)$ is the hypergeometric function and c, d are the rising factorial.

$${}_pF_q(c;d;x) = \sum_{k=0}^{\infty} \frac{(c_1)_k(c_2)_k \cdots (c_p)_k(x)^k}{(d_1)_k \cdots (d_q)_k k!}.$$

By the previous theorem and some identities of the Euler Γ function we find that

$$\mathcal{M}J_\nu(x) = \frac{2^{s-1}\Gamma((s+\nu)/2)}{\Gamma((\nu-s)/2+1)}.$$

Proposition 2.2. *The Parseval formula. Suppose f and g are such that $I = \int f(x)g(x)dx$ exists. Choose $Re(s) = c$ in the common strip of analycity of $\mathcal{M}[f;1-s]$ and $\mathcal{M}[g;s]$. Then*

$$I = \int_0^\infty f(x)g(x)dx = \frac{1}{2\pi i}\int_{Re(s)=c} \mathcal{M}[f;1-s]\mathcal{M}[g;s]ds$$

This provides another powerful tool for working with integrals of this type.

Proof. Using the definition of the Mellin transform, we have

$$\frac{1}{2\pi i}\int_{Re(s)=c} \mathcal{M}[f;1-s]\mathcal{M}[g;s]ds = \frac{1}{2\pi i}\int_{Re(s)=c} \mathcal{M}[f;1-s]\int_0^\infty g(t)t^{s-1}dtds.$$

By Fubini's theorem, after exchanging the order of the integrals we find

$$\frac{1}{2\pi i}\int_{Re(s)=c} \mathcal{M}[f;1-s]\mathcal{M}[g;s]ds = \frac{1}{2\pi i}\int_0^\infty g(t)\int_{Re(s)=c} \mathcal{M}[f;1-s]t^{s-1}dtds,$$

and using the inversion formula for f, we have the above Parseval relation. □

Within the theory of Mellin transforms, cutoff functions are used to ensure the convergence of the integrals involved. The idea is to multiply a function that decays rapidly at infinity, thereby reducing the contribution of large values of x to the integral. There are several choices of weight/cutoff functions which usually depend on the specific problem being considered and the properties of the function being transformed. Below are some common examples from the book "Tables of Mellin transforms"[Obe74]*:*

$f(x)$	$\mathcal{M}[f;s]$	*Strip of convergence*
e^{-x}	$\Gamma(s)$	$0 < \Re(s) < \infty$
$e^{-(\ln x)^2}$	$\sqrt{\pi}e^{\frac{s^2}{4}}$	$\Re(s) \in \mathbb{R}$
$\delta(x-a)$	a^{s-1}	$\Re(s) \in \mathbb{R}$
$\frac{1}{x+1}$	$\frac{\pi}{\sin(\pi s)}$	$0 < \Re(s) < 1$
$\ln(x+1)$	$\frac{\pi}{s\sin(\pi s)}$	$-1 < \Re(s) < 0$
$\sin(x)$	$\sin(\frac{\pi s}{2})\Gamma(s)$	$-1 < \Re(s) < 1$
e^{ix}	$e^{i\pi s/2}\Gamma(s)$	$0 < \Re(s) < 1$

In practice, the inverse Mellin transform is often evaluated numerically using numerical integration methods. For example, the following function in Pari/GP[The22]*:*

Example. ***infuncinit(t=a,b,f,{m=0})***. *Initializes tables for use with integral transforms (such as Fourier,Laplace or Mellin) in order to compute*

$$\int_a^b f(t)k(t,z)dt$$

for some kernel $k(t,z)$.

3. Occurences in the wild

The Mellin transform has been used in a wide range of research areas, including number theory, probability theory, and mathematical physics. In number theory, the Mellin transform is used to study the behavior of certain arithmetic functions, such as the Riemann zeta function. In probability theory, it is used to study the behavior of random variables and to derive limit theorems. In mathematical physics, it has been used to study problems related to wave propagation, scattering, and quantum field theory (QFT).

- *The analytic continuation of the zeta function to the entire complex plane and its functional equation have been derived using the Mellin transform. The proof relies heavily upon looking at the Mellin transform of the theta function*

  $$\theta(s) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 s} \quad \Re(s) > 0$$

  By applying the Mellin transform to the theta function, we recover the Riemann zeta function up to some scaling and correction terms.
- *More generally, the mellin transform has been used to study the properties of L-functions of objects such as modular and automorphic Forms. The transform is a big tool in understanding the interplay between the form and its L-function. It h has been used to study the Fourier coefficients of modular forms. For example, Hecke*[Hec] *gave conditions for a Dirichlet series to be the Mellin transform of a modular form. Questions of the form "does there exist an object with this specified L-function" and vice-versa are central in number theory - the modularity theorem and the Langlands programme are of note.*
- *Whittaker functions appear in the generic Fourier coefficients of cuspidally induced Eisenstein series. The Mellin transform of these functions is used to study their properties, such as their analytic continuation and growth estimates. For example, in the work of Langlands and Shahidi.* [Sha78]
- *The Selberg trace formula relates the trace of certain operators on a group to a sum over periodic orbits. We point the reader to the book of J. Arthur* [Art05]
- *In QFT, the operator product expansion (OPE) expresses the product of two local operators in terms of a sum over a complete set of operators with definite scaling dimensions. The OPE and its Mellin transform have many applications in the study of the renormalization group, conformal field theory, and the Anti deSitter/Conformal field theory (AdS/CFT) correspondence.*

- *Feynman integrals are integrals that arise in the perturbative expansion of scattering amplitudes in QFT. The Mellin transform of Feynman integrals has many applications in the study of the infrared and ultraviolet behavior of QFT, and in the development of efficient algorithms for their numerical evaluation.*

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Andreas Hatziiliou - Department of Mathematics, University of British Columbia
Email address: ahatzi@math.ubc.ca

Arnab Bose - Department of Mathematics and Computer Science, University of Lethbridge
Email address: arnab.bose@uleth.ca